ON ENERGY LANSCAPES
OF ELASTIC MANIFOLDS IN RANDOM POTENTIALS

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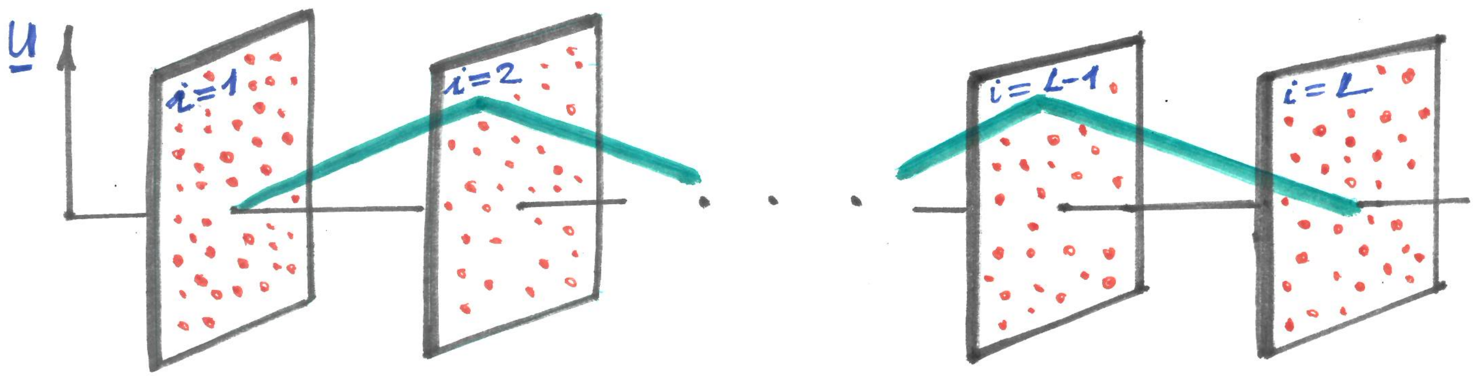
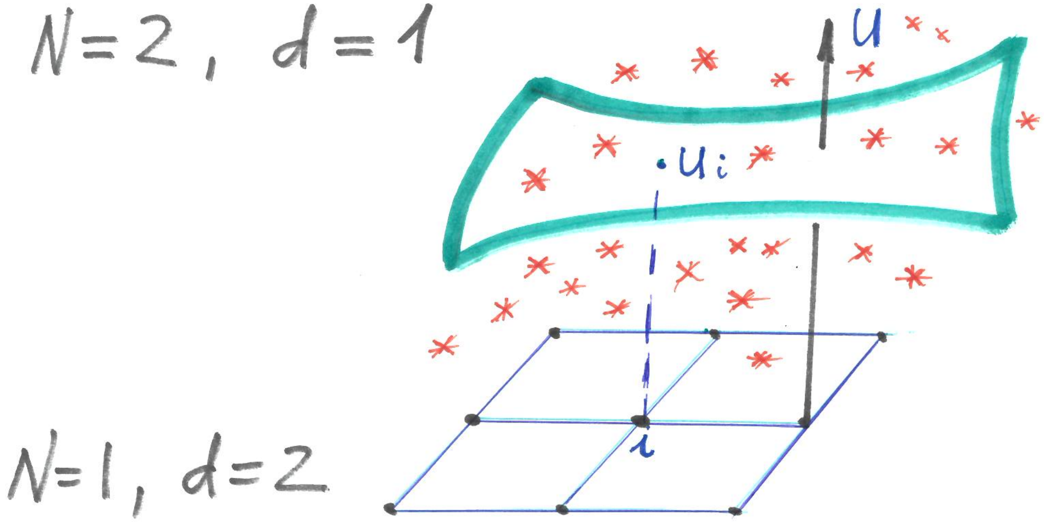
Elastic d-manifolds in random media of N+d dimensions.:





$$\mathcal{H}(\mathbf{u}_1,\ldots,\mathbf{u}_K)=\sum_{i=1}^{K=L^d}\left[\frac{\mu^2}{2}\mathbf{u}_i^2+V_i(\mathbf{u}_i)\right]-\frac{t}{2}\sum_{i\sim j}(\mathbf{u}_i-\mathbf{u_j})^2,\quad \mathbf{u_i}\in\mathbb{R}^{\mathbf{N}}$$

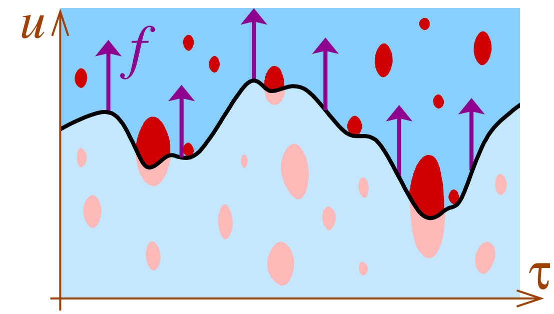
Content:

- **Part I:**
  Energy landscape and depinning for a directed polymer in a random potential.
  Based on: **YVF**, P. Le Doussal, A. Rosso, and C. Texier. *Ann. Phys.* **397**, 1–64 (2018).

- **Part II:**
  Mean-field limit of elastic manifolds:
  Hessian of energy landscape, complexity and depinning
  Based on: **YVF**, P. Le Doussal *arXiv: 1903.07159* and *under preparation*

Pierre Le Doussal **Alberto Rosso** **Christophe Texier**

Part I: Directed polymer in a random potential



We consider the following energy functional

$$\mathcal{H}[\mathbf{u}(\tau)] = \int_0^L d\tau \left[\frac{m^2}{2} [\mathbf{u}(\tau)]^2 + \mathbf{V}(\mathbf{u}(\tau), \tau) + \frac{\kappa}{2} \left(\frac{\partial \mathbf{u}(\tau))}{\partial \tau} \right)^2 - \mathbf{f}(\tau) \mathbf{u}(\tau) \right]$$

where $\mathbf{u}(\tau)$, $\tau \in [0, L]$ describes the polymer configuration trajectory, $\kappa \geq 0$ is the elastic energy coefficient, and $\mathbf{f}(\tau)$ is the external (depinning) force. We may assume the periodic/fixed ends configurations $u(0) = u(L)(= 0)$ for simplicity.

The random potential $\mathbf{V}(u(\tau), \tau)$ is chosen to be Gaussian with zero mean and with a translationally-invariant covariance

$$\overline{\mathbf{V}(u, \tau) \mathbf{V}(u', \tau')} = \delta(\tau - \tau') R(u - u')$$

where we assume the symmetric function $R(u)$ to be at least four times differentiable at $u = 0$. The notation $\overline{\cdots}$ stands for the quantities averaged over the random potential.

To have a better defined problem, the polymer is considered to be confined inside a harmonic well of curvature $m^2 \geq 0$, called the mass parameter, which flattens the line beyond an 'infrared' length, defined as

$$L_m := \sqrt{\frac{\kappa}{m}}$$

The limit $m \to 0^+$ is of special interest, as the system becomes **critical**, with a non-trivial roughness exponent arising as long as $L_m \to \infty$.
Potential disorder induces yet another characteristic scale:

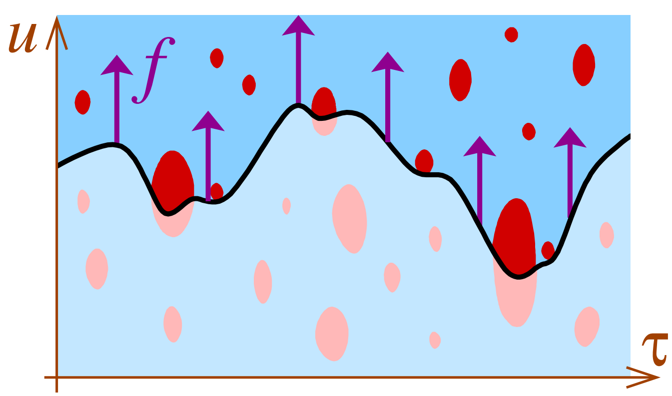
$$L_c := \left(\frac{\kappa^2}{R''''(0)}\right)^{1/3}$$ **Larkin** length.

When $m \to 0$ and $L \gg L_c$ the **metastability** effects due to pinning by multiple extrema become important, inducing line roughness.

Our goal: To characterize the **depinning threshold** f_c for $m \to 0$ and $L \to \infty$.

Our answer: $f_c \leq \sqrt{2C\frac{|R''(0)|}{L_c}} = f_c^*$, with $C \approx 0.46$

Numerical simulations for $R(u) = \sigma^2 \cos u$ give $f_c^* \approx 1.64 f_c$.



Anatoly Larkin (1932-2005)

Outline of the main results:

(**i**). For a given length L and a (uniform) force $f(\tau) = f$ we count all **energy functional equilibria** defined as solutions of $\frac{\delta \mathcal{H}[u(\tau)]}{\delta u} = 0$. Those include **minima**, **saddles**, and **maxima**. The number $\mathcal{N}(L,f)$ of equilibria is random, and we show its mean value behaves asymptotically as

$$\overline{\mathcal{N}(L,f)} \sim e^{L\left(r - \frac{f^2}{2|R''(0)|}\right)} \quad \text{as long as} \quad L \to \infty$$

We see that for $f > f_c^* = \sqrt{2r|R''(0)|}$ the mean value $\langle \mathcal{N}(L,f) \rangle$ is exponentially small, hence **no equilibria** exists in a typical realization providing an upper bound.

(**ii**) In fact we consider a more general problem and show that for $m > 0$

$$\overline{\mathcal{N}(L,m;f=0)} \sim e^{rL} \text{ as long as } L \to \infty$$

where the growth rate r is given by

$$r = \frac{1}{L_m} g\left(\frac{L_m}{L_c}\right)$$

and the scaling function $g(x)$ can be explicitly calculated in several limits, and is argued to be **universal**.

$$g(x) \approx \begin{cases} Cx, \quad x = L_m/L_c \gg 1, \quad C \approx 0.46 \\ \frac{1}{8\pi} x^3 \exp - \frac{8}{3x^2}, \quad x = \frac{L_m}{L_c} \to 0 \end{cases}$$

Counting zeroes via Kac-Rice formula:

Mark Kac (1914-1984) and **Stephen O. Rice** (1907-1986)

Number $\mathcal{N}_{(a,b)}$ of simple zeroes of a (smooth enough) function $f(x)$ in $x \in (a,b)$ can be found via

$$\mathcal{N}_{(a,b)} = \int_a^b \delta(f(x))|f'(x)|\,dx$$

Counting equilibria for the "discretized" model:

It is more convenient to start from a discrete version of the model, passing to the continuous limit in the end of calculations. In this way we replace the continuous variable τ by a discrete **lattice index** $i = 1, \ldots, K$ with $K = L/a$ (for simplicity we choose units such that the lattice spacing is $a = 1$). The energy of the polymer and the correlations of the random potential in such setting are given by

$$\mathcal{H}(\mathbf{u}) = \sum_{i=1}^{K} \left[\frac{m^2}{2} u_i^2 + V_i(u_i)\right] + \frac{\kappa}{2} \sum_{i=0}^{K} (u_i - u_{i+1})^2 \quad (1)$$

$$\overline{V_i(u) V_j(u')} = \delta_{ij}\, R(u - u')\,. \quad (2)$$

Configurations of the polymer are described by vectors of transverse coordinates $\mathbf{u}^T = (u_1, \ldots, u_K)$, with $u_i \in \mathbb{R}$ for $i = 1, 2, \ldots, K$. The periodic/fixed ends condition now read $u_0 = u_{K+1} (= 0)$. The effect of an external force field requires to add a term $-\mathbf{f}^{\mathrm{T}}\mathbf{u} = \sum_{i=1}^{K} f_i u_i$ to the energy.

Counting equilibria for the "discretized" model:

An equilibrium configuration is found as a solution of the system of K stationarity conditions which can be conveniently written as

$$\partial_i \mathcal{H}(\mathbf{u}) = \left[(m^2 \mathbf{I}_K - \kappa\Delta)\mathbf{u}\right]_i + V_i'(u_i) = 0\,, \quad i = 1, \ldots, K$$

where $\partial_i \equiv \partial/\partial u_i$ is the partial derivative, $\mathbf{I}_K$ is the identity matrix of size K and Δ is the **discrete Laplacian matrix** for the underlying one-dimensional lattice, with the only non-zero entries (e.g. for fixed ends boundary conditions) being

$$\Delta_{i,i} = -2\,, i = 1, \ldots, K \text{ and } \Delta_{i,i-1} = \Delta_{i-1,i} = +1$$

In the continuum limit such a matrix approximates the standard one-dimensional Laplacian operator $\frac{d^2}{d\tau^2}$ with Dirichlet boundary conditions.

The total number of solutions $\mathcal{N}_A$ of such equations such that $\mathbf{u}$ belongs to a subset A of $\mathbb{R}^K$ is then given by the multidimensional **Kac-Rice** formula:

$$\mathcal{N}_A = \int_A \rho(\mathbf{u}) d\mathbf{u} \quad \text{where} \quad \rho(\mathbf{u}) = \left|\det\left(\partial_i \partial_j \mathcal{H}\right)\right| \prod_{i=1}^{K} \delta\left(\partial_i \mathcal{H}\right)$$

with the Hessian being a $K \times K$ matrix given explicitly by

$$\partial_i \partial_j \mathcal{H} = \left[m^2 + V_i''(u_i)\right] \delta_{i,j} - \kappa\, \Delta_{i,j}\,. \tag{3}$$

Counting equilibria for the "discretized" model:

$$\mathcal{N}_A = \int_A \rho(\mathbf{u})d\mathbf{u} \quad \text{where} \quad \rho(\mathbf{u}) = |\det(\partial_i\partial_j\mathcal{H})| \prod_{i=1}^{K} \delta(\partial_i\mathcal{H})$$

To perform the disorder average we will use that:

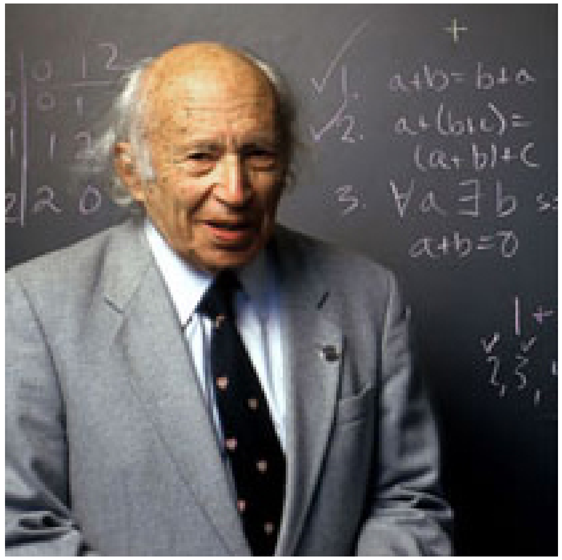
(a) the potentials $V_i(u)$ and $V_j(u)$ are statistically independent for $i \neq j$
(b) the variables $V_i'(u)$ are independent of $V_i''(u)$ for any i as an important consequence of translational invariance and the Gaussian character of the random function $V_i(u)$.

Moreover, after taking the average the mod-Hessian factor is obviously independent of $\mathbf{u}$, and the average of each of the K $\delta-$ factors can be done independently over the distribution of the Gaussian variable $V_i'(u)$ with the variance $\overline{[V_i'(u)]^2} = -R''(0)$. The corresponding Gaussian integral yields the constant Jacobian factor $|\det\left(m^2\,\mathbf{I}_K - \kappa\,\Delta\right)|^{-1}$ finally implying that

$$\overline{\mathcal{N}_{\text{tot}}} = \frac{\overline{\left|\det\left((m^2/\kappa)\,\delta_{ij} - \Delta_{ij} + U_i\,\delta_{ij}\right)\right|}}{\left|\det\left((m^2/\kappa)\,\delta_{ij} - \Delta_{ij}\right)\right|},$$

where the averaging goes over the set of i.i.d. mean-zero Gaussian random variables $U_j \equiv V_j''(u_j)/\kappa$ with the covariance structure $\langle U_i U_j\rangle = 2D\,\delta_{ij}$, where the parameter $D = \frac{R''''(0)}{2\kappa^2} = \frac{1}{2L_c^3}$, measures the strength of the disorder in the problem, and is directly related to the **Larkin length** at $m = 0$.

Evaluating determinants via Gelfand-Yaglom formula:



Israel Gelfand (1913-2009) and **Akiva Yaglom** (1921-2007)

Related determinants of 1D Schroedinger operators to solving an initial value problem.

Counting equilibria via functional Kac-Rice and Gelfand-Yaglom formulae:

$$\overline{\mathcal{N}(L)} = \int \overline{\left|\det \frac{\delta^2 \mathcal{H}}{\delta u \delta u'}\right| \delta\left(\frac{\delta \mathcal{H}}{\delta u}\right)} \mathcal{D}u = \frac{\overline{|\det (m^2 I + \kappa \mathbf{H})|}}{\det (m^2 I + \kappa \mathbf{H}_0)}$$

where $\mathbf{H}$ is a **random Schrödinger** operator with the 'white-noise' potential:

$$\mathbf{H} = -\frac{d^2}{d\tau^2} + U(\tau), \quad \overline{U(\tau_1)U(\tau_2)} = 2D\delta(\tau_1 - \tau_2)$$

with $D = \frac{R''''(0)}{2\kappa^2} \equiv \frac{1}{2L_c^3}$ and $\mathbf{H}_0 = -\frac{d^2}{d\tau^2}$.

Gelfand-Yaglom:

Consider $y(\tau)$ to be the solution of an Initial Value Problem:

$$\mathbf{H}y = -\frac{m^2}{\kappa} y, \quad y(0) = 0, \quad y'(0) = 1$$

Then
$$\overline{\mathcal{N}(L)} = \frac{\overline{|y(L)|}}{y_0(L)}$$

where $y_0(\tau)$ is the solution of a similar IVP for $\mathbf{H}_0$.

As is well-known, such IVP is characterized by an **exponential growth** of the solution in every realization, characterized by the **Lyapunov exponent**:

$$\gamma_1 = \lim_{L\to\infty} \frac{1}{L} \ln |y(L)| > 0$$

Statistics of $|y(L)|$:

Although the Lyapunov exponent is non-random (self-averaging) as $L \to \infty$, the fluctuations of $|y(L)|$ for $L \gg 1$ are extremely strong, and characterized by the Large Deviation type of expression:

$$\overline{|y(L)|^q} \sim \exp L\Lambda(q), \qquad \Lambda(q) := \lim_{L\to\infty} \frac{1}{L} \ln \overline{|y(L)|^q} = \sum_{n=1}^{\infty} \frac{\gamma_n}{n!} q^n$$

We need $\Lambda(1)$ as $r = \Lambda(1) - 1/L_m$. We compute it by mapping the Sch.Eq. to a stochastic differential equation for the **Ricatti** variable $z(\tau) = [y(\tau)]^{-1} \frac{dy}{d\tau}$, so that

$$\frac{dz}{d\tau} = m^2/\kappa - z^2 + U(\tau), \quad \overline{U(\tau_1)U(\tau_2)} = 2D\delta(\tau_1 - \tau_2)$$

and

$$\overline{|y(L)|^q} = \overline{\exp\left(q \int_0^L z(t)dt\right)}$$

The probability density $\mathcal{P}(z,\tau)$ satisfies the Fokker-Planck-Kolmogorov eqn:

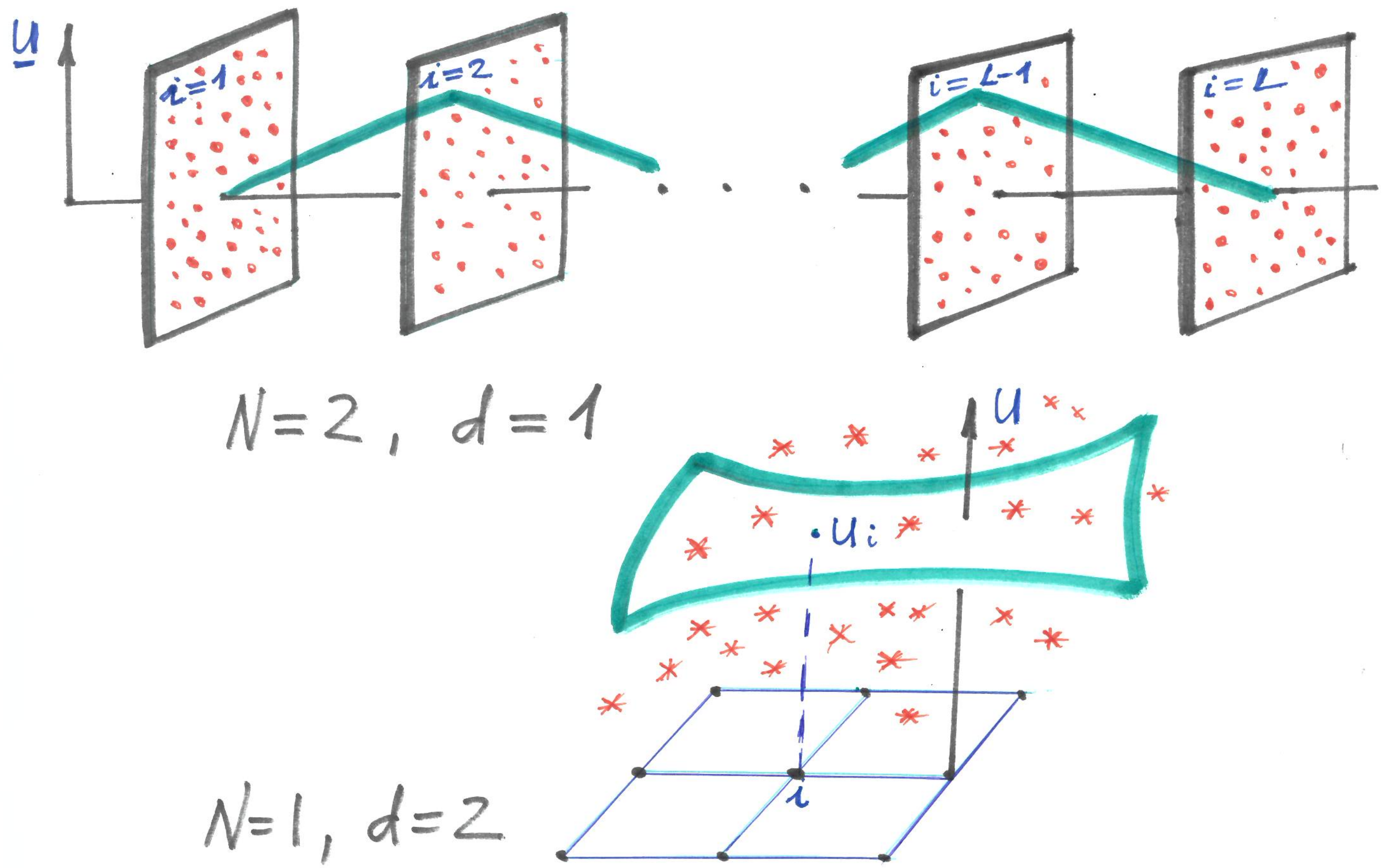
$$\frac{\partial}{\partial t}\mathcal{P} = \hat{S}\mathcal{P}, \quad \hat{S} := \frac{d}{dz}\left(m^2/\kappa - z^2 + D\frac{d}{dz}\right)$$

and one can show that

$$\overline{|y(L)|^q} = \int < z|e^{L\left(\hat{S}+qz\right)}|\infty > \, dz$$

The largest eigenvalue of $\mathcal{L} = \hat{S} + qz$ gives $\Lambda(q)$.

Elastic d-manifolds in random media of N+d dimensions.:



$$\mathcal{H}(\mathbf{u}_1,\ldots,\mathbf{u}_K)=\sum_{i=1}^{K=L^d}\left[\frac{\mu^2}{2}\mathbf{u}_i^2+V_i(\mathbf{u}_i)\right]-\frac{t}{2}\sum_{i\sim j}(\mathbf{u}_i-\mathbf{u_j})^{\mathbf{2}},\quad \mathbf{u_i}\in\mathbb{R}^{\mathbf{N}}$$

Part II. Hessians for manifolds in random media:

The manifold is parameterized by a N-component real displacement field $\mathbf{u}(x) \in \mathbb{R}^N$ where $x \in L^d \subset \mathbb{Z}^d$, with the energy functional

$$\mathcal{H}[\mathbf{u}] = \sum_{x,y} \mathbf{u}(x) \cdot (\mu \mathbf{1} - t\Delta)_{xy} \cdot \mathbf{u}(y) + \sum_x V(\mathbf{u}(x), x)$$

where $t > 0$, and Δ is the discrete Laplacian in the hypercube L^d with periodic boundary conditions, eigenmodes $\sim e^{ikx}$ and eigenvalues $\Delta(k)$ (e.g. in $d = 1$, $\Delta(k) = 2(\cos k - 1)$ with $k = 2\pi n/L$, $n = 0, ..L - 1$). We will eventually consider the limit of the continuum manifold with the standard Laplacian $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ whose spectrum is given by $\Delta(\mathbf{k}) = -\mathbf{k}^2$.

The potential $V(\mathbf{u}, x)$ is Gaussian-distributed random potential in $\mathbb{R}^N \times \mathbb{Z}^d$ with mean zero and covariance

$$\overline{V(\mathbf{u}_1, x_1)V(\mathbf{u}_2, x_2)} = N\, B\left(\frac{(\mathbf{u}_1 - \mathbf{u}_2)^2}{N}\right) \delta^d(x_1 - x_2),$$

such that potential values are uncorrelated for different points in the internal space, but correlated for different displacements.

As usual, μ acts as a “mass" which, for the continuum model, leads to reducing the fluctuations beyond the scale $L_\mu = \sqrt{t/\mu}$.

We will only consider the mean-field type limit: $N \gg 1$.

Manifold Hessian, discrete lattice:

Our first goal will be to study the mean eigenvalue density $\rho(\lambda)$ for $NL^d \times NL^d$ Hessian matrix

$$K_{ix,jy}[\mathbf{u}] = \frac{\partial^2}{\partial u_i(x)\partial u_j(y)}\mathcal{H}[\mathbf{u}] = \delta_{ij}(\mu\mathbf{1} - t\Delta)_{xy} + \delta_{xy}\frac{\partial^2}{\partial u_i \partial u_j}V(\mathbf{u}(x), x)$$

of the pinned elastic manifold. An important feature of such matrix is its (periodic) block-band structure visualized below for $d = 1$:

$$\begin{bmatrix} \mathbf{X}_N^{(1)} & -t\mathbf{1}_N & 0 & \ldots & 0 & -t\mathbf{1}_N \\ -t\mathbf{1}_N & \mathbf{X}_N^{(2)} & -t\mathbf{1}_N & 0 & \ldots & 0 \\ 0 & -t\mathbf{1}_N & \mathbf{X}_N^{(3)} & -t\mathbf{1}_N & 0 & \ldots \\ \ldots & 0 & -t\mathbf{1}_N & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & -t\mathbf{1}_N & \mathbf{X}_N^{(L-1)} & -t\mathbf{1}_N \\ -t\mathbf{1}_N & 0 & \ldots & 0 & -t\mathbf{1}_N & \mathbf{X}_N^{(L)} \end{bmatrix}$$

where we introduced $N \times N$ diagonal blocks

$$\mathbf{X}_N^{(r)} := (\mu + 2t)\mathbf{1}_N + \mathbf{W}^{(r)},\; r = 1, \ldots, L$$

containing random matrices with entries

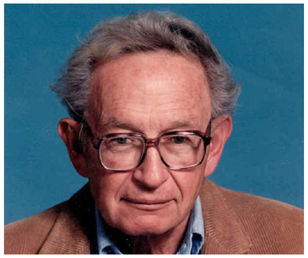
$$\mathbf{W}_{ij}^{(r)} = \frac{\partial^2}{\partial u_i \partial u_j}V(\mathbf{u}(x), x)\Big|_{x=x_r}$$

Manifold Hessian, statistics:

If the Hessian is chosen at a generic point in configuration space, i.e. at an arbitrary *fixed* $\mathbf{u}(x)$ the statistical translational invariance of the random potential implies the Hessian matrix is statistically independent of the choice of $\mathbf{u}(x)$, i.e. we may as well chose it at $\mathbf{u}(x) = \mathbf{0}$. The covariance structure of the random potential further implies that entries of the matrices $\mathbf{W}^{(r)}$ are mean-zero Gaussian-distributed, independent for different r and have the following covariance structure:

$$\overline{\mathbf{W}_{ij}^{(r)}\mathbf{W}_{kl}^{(s)}} = \delta_{rs}\frac{4}{N}B''(0)\left(\delta_{ij}\delta_{lk} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

The matrices of such block-band type, with $\mathbf{W}^{(r)}$ in diagonal blocks replaced with GOE matrices with i.i.d. entries, were introduced by **Franz Wegner** in his famous studies of the **Anderson localization**, and are now known by the general name of Wegner **orbital** models.



Philip Anderson

Franz Wegner

Manifold Hessian in the continuum:

In the case of a continuous manifold the Hessian matrix $\mathcal{K}$ becomes a **matrix-valued** differential operator $\mathcal{K}$ acting in the space of $N-$component vectors $\mathbf{f}(x) := (f_1(x), \ldots, f_N(x))^T$ where, e.g. $x \in [0, L]^d$, by the following rule:

$$\mathcal{K}\mathbf{f} = (\mu\mathbf{1} - t\Delta)\mathbf{f} + \hat{W}\mathbf{f}, \quad W_{i,j}(x) = \frac{\partial^2}{\partial u_i \partial u_j} V(\mathbf{u}(x), x)$$

with appropriate boundary conditions. In particular, for $d = 1$ the operator $\mathcal{K}$ can be visualized in the following form of an $N \times N$ matrix:

$$\begin{pmatrix} -t\frac{d^2}{dx^2} + \mu + W_{1,1}(x) & W_{1,2}(x) & \ldots & W_{1,N}(x) \\ W_{1,2}(x) & -t\frac{d^2}{dx^2} + \mu + W_{2,2}(x) & \ldots & W_{2,N}(x) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ W_{1,N}(x) & \ldots & W_{N,N-1}(x) & -t\frac{d^2}{dx^2} + \mu + W_{N,N}(x) \end{pmatrix}$$

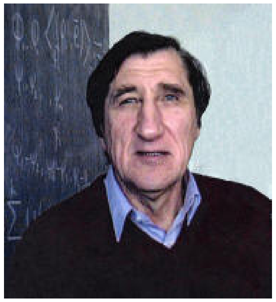
where

$$\overline{W_{i,j}(x_1)W_{k,l}(x_2)} = \delta(x_1 - x_2)\frac{4}{N}B''(0)\left(\delta_{ij}\delta_{lk} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

Models of such type are sometimes called the **matrix Anderson** models.

We will show that the associated profiles of the mean eigenvalue density for such problem can be explicitly found as long as $L \to \infty$ *after* $N \to \infty$.

Manifold Hessian's resolvent:



Leonid Pastur

Our main object of interest will be the disorder-averaged resolvent (Green's function) of the Hessian, calculated at the **global minimal energy** configuration $\mathbf{u}_0 \equiv \mathbf{u}_0(x)$.:

$$\overline{\mathcal{G}(x,y;\lambda,\mathbf{u}_0)} = \frac{1}{N}\sum_{i=1}^{N} \overline{\left(\frac{1}{\lambda - \mathcal{K}(\mathbf{u}_0)}\right)_{xi,yi}}$$

Employing the **replica trick**, we first show that for $N \to \infty$ (the limit being taken for a fixed value of L^d) the average Green's function is given by

$$\overline{G(x,y;\lambda,\mathbf{u}_0)} = \int_k \frac{e^{ik(x-y)}}{\lambda - \mu_{\text{eff}} + t\Delta(k) - 4ipB''(0)}$$

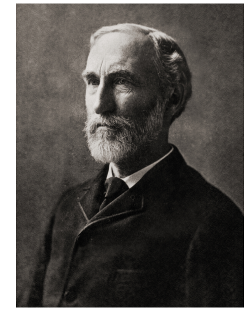
where the value of the parameter p is determined by the following self-consistent '**deformed semicircle**'/**Pastur** equation for the diagonal part

$$\overline{G(x,x;\lambda,\mathbf{u}_0)} = ip = \int_k \frac{1}{\lambda - \mu_{\text{eff}} + t\Delta(k) - 4ipB''(0)}$$

which is essentially of the same form as one for the orbital model with lattice Laplacian (**Khorunzhii** $\&$ **Pastur** '93).

The only quantity which contains all the information about the optimization leading to the ground state $\mathbf{u}_0$ is the parameter μ_{eff}. Below μ_{eff} will be calculated in the various cases (replica-symmetric, 1RSB and FRSB) in the framework of the replica theory.

Addressing the global minimum via Statistical Mechanics:



J W Gibbs (1839–1903)

Given the **energy function(al)** $\mathcal{H}[\mathbf{u}]$ associated with a manifold configuration $\mathbf{u}(x)$ we introduce the inverse temperature parameter $\beta > 0$, and define the partition function of the model as the (functional) integral

$$\mathcal{Z}_\beta = \int e^{-\beta\mathcal{H}[\mathbf{u}]} D\mathbf{u}.$$

We further define the (Boltzmann-)**Gibbs** weights $\pi_\beta[\mathbf{u}(x)] = \mathcal{Z}_\beta^{-1} e^{-\beta\mathcal{H}[\mathbf{u}]}$ associated with any manifold configuration $\mathbf{u}(x)$ and define the thermal averaged value of any functional $g[\mathbf{u}]$ as $\langle g[\mathbf{u}]\rangle_\beta := \int g[\mathbf{u}]\pi_\beta[\mathbf{u}]D\mathbf{u}$.

In the zero-temperature limit $\beta \to \infty$ the Gibbs weights concentrate on the set of configurations delivering the global minimum $\mathbf{u}_0(x) = \mathrm{Argmin}\,\{\mathcal{H}[\mathbf{u}(x)]\}$ so that for any well-behaving functional

$$\lim_{\beta\to\infty}\langle g[\mathbf{u}]\rangle_\beta = g(\mathbf{u}_0).$$

Although this fact is valid in every disorder realization, in practice one concentrates on finding the disorder-averaged values $\overline{\langle g(\mathbf{u})\rangle_\beta}$. In particular, in this talk we choose the function $g(\mathbf{u})$ as the resolvent of the Hessian.

Spectral Density of the Hessian at a generic point:

With setting $\mu_{eff} = \mu$ the above expressions provides the mean resolvent and the mean spectral density $\rho(\lambda)$ for the manifold Hessian around a generic point of the disordered landscape.

Its generic feature is the square-root singularity at the spectral edges, which is thus a universal characteristics of the mean-field type spectral densities for disordered elastic systems of any dimension d. The shape as a whole is not universal and essentially depends on the dimension and the type of the Laplacian matrix (discrete or continuous). Interestingly, it turns out to be possible to find explicitly the spectral density for the 1D matrix Anderson model of infinite length $L \to \infty$ and the Laplacian spectrum $-\Delta(k) = k^2$, $-\infty < k < \infty$:

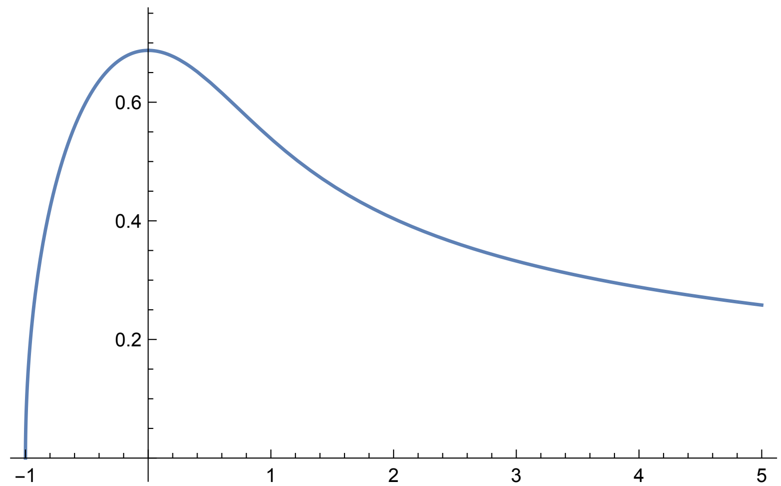
$$\rho(\lambda) = \frac{1}{2\pi (t\, B''(0))^{1/3}}\, r_c\left(\Lambda = t^{1/3} \frac{\lambda - \mu}{3B''(0)^{2/3}}\right), \quad r_c(\Lambda) = \frac{w_r^2}{4}\sqrt{\left(\frac{2}{w_r}\right)^3 - 1}$$

where

$$w_r = \left[1 + \sqrt{1+\Lambda^3}\right]^{1/3} + \left[1 - \sqrt{1+\Lambda^3}\right]^{1/3},$$

The parameter free scaling function $r_c(\Lambda)$ is plotted below.

Spectral Density of the Hessian in 1D continuum case:



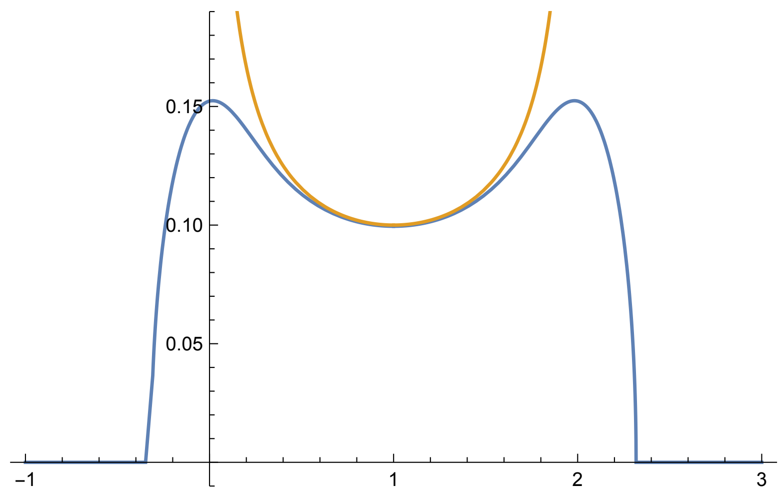
Scaling function $r_c(\Lambda)$ for the Hessian spectral density for the $d = 1$ continuum model plotted versus $\Lambda = t^{1/3}\frac{\lambda - \mu_{\text{eff}}}{3B''(0)^{2/3}}$. The spectral edge Λ_e is given in this case by $\Lambda_e = -1$. The function $r_c(\Lambda)$ reaches its maximum at $\Lambda = 0$ and then decays at $\Lambda \gg 1$ as $r_c(\Lambda \gg 1) \sim \frac{1}{\sqrt{3\Lambda}}$. The latter regime corresponds to the spectral density $\rho(\lambda) = \frac{1}{2\pi}\frac{1}{\sqrt{t(\lambda-\mu)}}$ of the disorder-free operator $\mu - \frac{d^2}{dx^2}$ with the spectrum $\lambda = \mu + tk^2$.

Spectral Density of the Hessian in 1D discrete lattice:

For 1D elastic discrete chain with Laplacian spectrum $-\Delta(k) = 2(1-\cos k), 0 \leq k \leq 2\pi$ **the shape of the spectral density for the associated disordered banded Hessian can be shown to be of the form**

$$\rho(\lambda) = \frac{t}{2\pi B''(0)} r\left(\Lambda = \frac{\lambda-\mu}{2t}, y\right)\ , y = \frac{t^2}{B''(0)}$$

but the function $r(\Lambda, y)$ **does not have a simple form, apart from the case of weak disorder** $y \gg 1$**.**



Blue: scaling function for the Hessian spectral density, $r(\Lambda, y)$ versus $\Lambda = \frac{\lambda-\mu}{2t}$ for $y = \frac{t^2}{B''(0)} = 10$ (weak disorder). In the weak disorder limit, the central part converges to the spectral density without disorder (indicated here in orange). The graph $r(\Lambda, y)$ has two spectral edges at $\Lambda_e^{(-)} = -\frac{3}{2}y^{-2/3}$ and $\Lambda_e^{(+)} = 2 + \frac{3}{2}y^{-2/3}$, and the two parts around the edges converge, upon rescaling, to the density for the continuum model.

Hessian spectrum at global energy minimum: Larkin mass and RS phase:

The most important parameter in the theory is the "**Larkin** mass" $\mu_c > 0$ which controls the value of the parabolic confinement μ below which the **replica symmetry breaking** (RSB) occurs at $T = 0$. Its value turns out to be given by the *positive* solution of

$$1 = 4B''(0)\int_k \frac{1}{(-t\Delta(k)+\mu_c)^2}$$

which is controlled both by disorder strength and the elasticity matrix. For example, for $1D$ continuous system a simple calculation gives $\mu_c = \left(\frac{B''(0)}{\sqrt{t}}\right)^{2/3}$.

Our analysis shows that in the **replica symmetric** phase

$$\mu_{eff} = \mu + 4B''(0)\int_k \frac{1}{\mu - t\Delta(k)}$$

and the **lower** spectral edge $\lambda_e^{(-)}$ of the Hessian (which we associate with the spectral gap) as a function of μ is given by

$$\lambda_e^{(-)} = \mu - \mu_c + 4B''(0)\int_k \left[\frac{1}{\mu - t\Delta(k)} - \int_k \frac{1}{\mu_c - t\Delta(k)}\right]$$

This formula immediately shows that for $\mu > \mu_c$ the Hessian spectrum is always gapped (from zero). Upon expanding for $\mu \to \mu_c$ one immediately finds the gap vanishing quadratically at μ_c.

Hessian spectrum at the point of global energy minimum, FRSB:

We consider random Gaussian potentials with the power-law class covariance function $B(q) = g^2(c+q)^{-(\gamma-1)}, \gamma > 0$, which also includes the (i) exponential $B(q) = g^2 e^{-Cq}$ as the limit $\gamma \to +\infty$, and (ii) the log-correlated case for $\gamma \to 1$, when $B(0) - B(q) = g^2 \ln\left(1 + \frac{q}{\epsilon}\right)$, where g and $\epsilon > 0$ are given constants. We also consider only the manifolds of dimenions $0 \le d < 4$ and $N \to \infty$.

As found in **Mezard and Parisi** '92 for $\mu < \mu_c$, **full** replica symmetry breaking, **FRSB**, always occurs for manifold of dimensions $2 < d < 4$, whereas for $0 \le d < 2$ FRSB occurs whenever $0 < \gamma < \frac{2}{2-d}$.

Our analysis shows that in the **full replica symmetry broken** phase the parameter μ_{eff} **freezes** to μ-independent value:

$$\mu_{eff} = \mu_c + 4B''(0) \int_k \frac{1}{\mu_c - t\Delta(k)}, \quad \forall \mu < \mu_c$$

and the lower spectral edge freezes at **zero**: $\lambda_e^{(-)} \equiv 0, \ \mu < \mu_c$.

Thus, the Hessian at the global minimum in the **Parisi-FRSB** phase of the pinned manifoiold is always **gapless**.

Giorgio Parisi

Hessian spectrum at the point of global energy minimum, 1-RSB:

In constrast, for $N \to \infty$ manifolds with $d < 2$ and $\gamma > \frac{2}{2-d}$ the **1-step** replica symmetry breaking occurs. In that case we find that the gap vanishes as $(\mu_c - \mu)^4$ near the transition from below, with the super-universal exponent. For example, for the continuum model in dimension d we get for $\mu = \mu_c(1-\delta)$

$$\lambda_e^{(-)} = \frac{\mu_c}{36 B^{(3)}(0)^4} \left(\frac{4-d}{4}\right)^3 \left(B^{(4)}(0)B''(0) - \frac{2(3-d)}{4-d} B^{(3)}(0)^2\right)^2 \delta^4 + O\left(\delta^5\right)$$

Special **marginal** cases share the properties of both **FRSB** and **1-RSB**. For continuous manifolds those are :

(**i**) $\gamma = 1$ for **d=0**: $B(0) - B(q) = g^2 \ln\left(1 + \frac{q}{\epsilon}\right)$

(**ii**) $\gamma = 0$ for **d=1**: $B(q) = \frac{g^2}{C+q}$

(**iii**) $\gamma = \infty$ for **d=2**: $B(q) = g^2 e^{-Cq}$

In particular, the **1-RSB** Parisi equations in all those cases can be exactly and explicitly solved, and the Hessian at the global minimum proves to be **gapless** for all curvatures below the Larkin mass.

Complexity, Larkin Length, and Depinning Threshold for $N \gg 1$:

(**i**). As $N \to \infty$ for a given L^d we can count all **equilibria** of the energy function(al) defined as solutions of $\frac{\delta \mathcal{H}[u(\tau)]}{\delta u} = 0$. We also can count separately all **stable equilibria** i.e. local **minima**. The numbers $\mathcal{N}(L)$ of all equilibria/minima are random, and we show the mean values behave asymptotically as

$$\overline{\mathcal{N}(L,\mu)} \sim e^{NL^d\,\Sigma_{tot}(\mu)}, \quad \overline{\mathcal{N}_m(L,\mu)} \sim e^{NL^d\,\Sigma_{st}(\mu)}$$

where the complexity $\Sigma_{st}(\mu)$ for minima is zero for $\mu > \mu_c$ whereas for $\mu < \mu_c$ given in terms of the **Larkin mass** μ_c as

$$\Sigma_{st}(\mu) = -\frac{1}{2}\left[\left(\mu_c - \mu + \int_k \frac{1}{\mu_c - t\Delta(k)}\right)^2 - \left(\int_k \frac{1}{\mu_c - t\Delta(k)}\right)^2 - \int_k \ln\frac{\mu - t\Delta(k)}{\mu_c - t\Delta(k)}\right]$$

(**ii**) From this one can show that $\Sigma_{st}(\mu \to \mu_c - 0) \sim (\mu - \mu_c)^3$ confirming the **third order** nature of the spinglass-type transitions. Such calculation also allows to naturally define the **Larkin length** so that $\overline{\mathcal{N}(L,\mu \to 0)} \sim e^{N\,(L/L_c)^d}$ giving $L_c = [\Sigma_{st}(0)]^{-\frac{1}{d}}$. After some calculation in the case of weak disorder $t \gg 1$ one gets the relation:

$$L_c \sim \left(\frac{t^2}{B''(0)}\right)^{\frac{1}{4-d}}$$

which nicely agrees with earlier estimates in the literature.

(**iii**) Finally, we can also include a (uniform) force $f(\tau) = f$ along a direction u_1 and show that the **depinning threshold** for $N = \infty$ model is given by

$$f_{st} = \sqrt{4B'(0)}\, L_c^{-d/2}$$

Moreover, as long as the underlying system is of 1-**RSB** nature we argue that this value is actually *exact*, i.e. valid for a **typical** disorder realizations.

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THANK YOU!